

Close-to-convexity and Starlikeness of Analytic Functions

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ABSTRACT. For functions $f(z) = z^p + a_{n+1}z^{p+1} + \dots$ defined on the open unit disk, the condition $\Re(f'(z)/z^{p-1}) > 0$ is sufficient for close-to-convexity of f . By making use of this result, several sufficient conditions for close-to-convexity are investigated and relevant connections with previously known results are indicated.

1. Introduction

Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk and $\mathcal{A}_{p,n}$ be the class of all analytic functions $f : \mathbb{D} \rightarrow \mathbb{C}$ of the form $f(z) = z^p + a_{n+p}z^{n+p} + a_{n+p+1}z^{n+p+1} + \dots$ with $\mathcal{A} := \mathcal{A}_{1,1}$. For studies related to multivalent functions, see [5, 7–10]. Singh and Singh [15] obtained several interesting conditions for functions $f \in \mathcal{A}$ satisfying inequalities involving $f'(z)$ and $zf''(z)$ to be univalent or starlike in \mathbb{D} . Owa *et al.* [11] generalized the results of Singh and Singh [15] and also obtained several sufficient conditions for close-to-convexity, starlikeness and convexity of functions $f \in \mathcal{A}$. In fact, they have proved the following theorems.

THEOREM 1.1. [11, Theorems 1-3] *Let $0 \leq \alpha < 1$ and $\beta, \gamma \geq 0$. If $f \in \mathcal{A}$, then*

$$\operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > \frac{1+3\alpha}{2(1+\alpha)} \implies \operatorname{Re} (f'(z)) > \frac{1+\alpha}{2},$$

$$\operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) < \frac{3+2\alpha}{(2+\alpha)} \implies |f'(z) - 1| < 1 + \alpha,$$

$$|f'(z) - 1|^\beta |zf''(z)|^\gamma < \frac{(1-\alpha)^{\beta+\gamma}}{2^{\beta+2\gamma}} \implies \operatorname{Re} (f'(z)) > \frac{1+\alpha}{2}.$$

THEOREM 1.2. [11, Theorem 4] *Let $1 < \lambda < 3$. If $f \in \mathcal{A}$, then*

$$\operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) < \begin{cases} \frac{5\lambda-1}{2(\lambda+1)}, & 1 < \lambda \leq 2; \\ \frac{\lambda+1}{2(\lambda-1)}, & 2 < \lambda < 3, \end{cases} \implies \frac{zf'(z)}{f(z)} \prec \frac{\lambda(1-z)}{\lambda-z}.$$

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In this present paper, the above results are extended for functions $f \in \mathcal{A}_{p,n}$ and in particular for functions in $\mathcal{A}_{1,n}$.

2. Close-to-convexity and Starlikeness

For $f \in \mathcal{A}$, the condition $\operatorname{Re} f'(z) > 0$ implies close-to-convexity and univalence of f . Similarly, for $f \in \mathcal{A}_{p,1}$, the inequality $\operatorname{Re}(f'(z)/z^{p-1}) > 0$ implies p -valency of f . See [17, 18]. From this result, the functions satisfying the hypothesis of Theorems 2.1–2.3 are p -valent in \mathbb{D} . A function $f \in \mathcal{A}_{p,1}$ is close-to-convex if there is a p -valent convex function ϕ such that $\operatorname{Re}(f'(z)/\phi(z)) > 0$. Also they are all close-to-convex with respect to $\phi(z) = z^p$.

THEOREM 2.1. *If the function $f \in \mathcal{A}_{p,n}$ satisfies the inequality*

$$(2.1) \quad \operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > \frac{(2p-n) + \alpha(2p+n)}{2(\alpha+1)},$$

then

$$\operatorname{Re} \left(\frac{f'(z)}{pz^{p-1}} \right) > \frac{1+\alpha}{2}.$$

For the proof of our main results, we need the following lemma.

LEMMA 2.1. [6, Lemma 2.2a] *Let $z_0 \in \mathbb{D}$ and $r_0 = |z_0|$. Let $f(z) = a_n z^n + a_{n+1} z^{n+1} + \dots$ be continuous on $\overline{\mathbb{D}}_{r_0}$ and analytic on $\mathbb{D}_{r_0} \cup \{z_0\}$ with $f(z) \neq 0$ and $n \geq 1$. If*

$$|f(z_0)| = \max\{|f(z)| : z \in \overline{\mathbb{D}}_{r_0}\},$$

then there exists an $m \geq n$ such that

- (1) $\frac{z_0 f'(z_0)}{f(z_0)} = m$, and
- (2) $\operatorname{Re} \frac{z_0 f''(z_0)}{f'(z_0)} + 1 \geq m$.

PROOF OF THEOREM 2.1. Let the function w be defined by

$$(2.2) \quad \frac{f'(z)}{pz^{p-1}} = \frac{1 + \alpha w(z)}{1 + w(z)}.$$

Then clearly w is analytic in \mathbb{D} with $w(0) = 0$. From (2.2), some computation yields

$$(2.3) \quad 1 + \frac{zf''(z)}{f'(z)} = p + \frac{\alpha zw'(z)}{1 + \alpha w(z)} - \frac{zw'(z)}{1 + w(z)}.$$

Suppose there exists a point $z_0 \in \mathbb{D}$ such that

$$|w(z_0)| = 1 \text{ and } |w(z)| < 1 \text{ when } |z| < |z_0|.$$

Then by applying Lemma 2.1, there exists $m \geq n$ such that

$$(2.4) \quad z_0 w'(z_0) = m w(z_0), \quad (w(z_0) = e^{i\theta}; \theta \in \mathbb{R}).$$

Thus, by using (2.3) and (2.4), it follows that

$$\begin{aligned}
 \operatorname{Re} \left(1 + \frac{z_0 f''(z_0)}{f'(z_0)} \right) &= p + \operatorname{Re} \left(\frac{\alpha m w(z_0)}{1 + \alpha w(z_0)} \right) - \operatorname{Re} \left(\frac{m w(z_0)}{1 + w(z_0)} \right) \\
 &= p + \operatorname{Re} \left(\frac{\alpha m e^{i\theta}}{1 + \alpha e^{i\theta}} \right) - \operatorname{Re} \left(\frac{m e^{i\theta}}{1 + e^{i\theta}} \right) \\
 &= p + \frac{\alpha m (\alpha + \cos \theta)}{1 + \alpha^2 + 2\alpha \cos \theta} - \frac{m}{2} \\
 &\leq \frac{(2p - n) + \alpha(2p + n)}{2(\alpha + 1)}
 \end{aligned}$$

which contradicts the hypothesis (2.1). It follows that $|w(z)| < 1$, that is

$$\left| \frac{1 - \frac{f'(z)}{p z^{p-1}}}{\frac{f'(z)}{p z^{p-1}} - \alpha} \right| < 1.$$

This evidently completes the proof of Theorem 2.1. \square

Owa [13] shows that a function $f \in \mathcal{A}_{p,1}$ satisfying $\operatorname{Re}(1 + z f''(z)/f'(z)) < p + 1/2$ implies f is p -valently starlike. Our next theorem investigates the close-to-convexity of this type of functions. For related results, see [4, 14, 19].

THEOREM 2.2. *If the function $f \in \mathcal{A}_{p,n}$ satisfies the inequality*

$$(2.5) \quad \operatorname{Re} \left(1 + \frac{z f''(z)}{f'(z)} \right) < \frac{(p+n)\alpha + (2p+n)}{(\alpha+2)},$$

then

$$\left| \frac{f'(z)}{p z^{p-1}} - 1 \right| < 1 + \alpha.$$

PROOF. Consider the function w defined by

$$(2.6) \quad \frac{f'(z)}{p z^{p-1}} = (1 + \alpha)w(z) + 1.$$

Then clearly w is analytic in \mathbb{D} with $w(0) = 0$. From (2.6), some computation yields

$$(2.7) \quad 1 + \frac{z f''(z)}{f'(z)} = p + \frac{(1 + \alpha)z w'(z)}{(1 + \alpha)w(z) + 1}.$$

Suppose there exists a point $z_0 \in \mathbb{D}$ such that

$$|w(z_0)| = 1 \text{ and } |w(z)| < 1 \text{ when } |z| < |z_0|.$$

Then by applying Lemma 2.1, there exists $m \geq n$ such that

$$(2.8) \quad z_0 w'(z_0) = m w(z_0), \quad (w(z_0) = e^{i\theta}; \theta \in \mathbb{R}).$$

Thus, by using (2.7) and (2.8), it follows that

$$\begin{aligned}
 \operatorname{Re} \left(1 + \frac{z_0 f''(z_0)}{f'(z_0)} \right) &= p + \operatorname{Re} \left(\frac{(1 + \alpha) z_0 w'(z_0)}{(1 + \alpha) w(z_0) + 1} \right) \\
 &= p + \operatorname{Re} \left(\frac{(1 + \alpha) m e^{i\theta}}{(1 + \alpha) e^{i\theta} + 1} \right) \\
 &= p + \frac{m(1 + \alpha)(1 + \alpha + \cos \theta)}{1 + (1 + \alpha)^2 + 2(1 + \alpha) \cos \theta} \\
 &\geq \frac{(p + n)\alpha + (2p + n)}{(\alpha + 2)},
 \end{aligned}$$

which contradicts the hypothesis (2.5). It follows that $|w(z)| < 1$, that is,

$$\left| \frac{f'(z)}{p z^{p-1}} - 1 \right| < 1 + \alpha.$$

This evidently completes the proof of Theorem 2.2. \square

Owa [12] has also showed that a function $f \in \mathcal{A}$ satisfying $|f'(z)/g'(z) - 1|^\beta |zf''(z)/g'(z) - zf'(z)g''(z)/(g'(z))^2|^\gamma < (1 + \alpha)^{\beta + \alpha}$, for $0 \leq \alpha < 1$, $\beta \geq 0$, $\gamma \geq 0$ and g a convex function, is close-to-convex. Also, see [3]. Our next theorem investigates the close-to-convexity of similar class of functions.

THEOREM 2.3. *If $f \in \mathcal{A}_{p,n}$, then*

$$(2.9) \quad \left| \frac{f'(z)}{p z^{p-1}} - 1 \right|^\beta \left| \frac{f''(z)}{z^{p-2}} - (p-1) \frac{f'(z)}{z^{p-1}} \right|^\gamma < \frac{(pn)^\gamma (1 - \alpha)^{\beta + \gamma}}{2^{\beta + 2\gamma}}$$

implies

$$\operatorname{Re} \left(\frac{f'(z)}{p z^{p-1}} \right) > \frac{1 + \alpha}{2},$$

and

$$(2.10) \quad \left| \frac{f'(z)}{p z^{p-1}} - 1 \right|^\beta \left| \frac{f''(z)}{z^{p-2}} - (p-1) \frac{f'(z)}{z^{p-1}} \right|^\gamma < (pn)^\gamma |1 - \alpha|^{\beta + \gamma}$$

implies

$$\left| \frac{f'(z)}{p z^{p-1}} - 1 \right| < 1 - \alpha.$$

PROOF. For the function w defined by

$$(2.11) \quad \frac{f'(z)}{p z^{p-1}} = \frac{1 + \alpha w(z)}{1 + w(z)},$$

we can rewrite (2.11) to yield

$$\frac{f'(z)}{p z^{p-1}} - 1 = \frac{(\alpha - 1)w(z)}{1 + w(z)}$$

, which leads to

$$(2.12) \quad \left| \frac{f'(z)}{pz^{p-1}} - 1 \right|^\beta = \frac{|w(z)|^\beta |1 - \alpha|^\beta}{|1 + w(z)|^\beta}.$$

By some computation, it is evident that

$$\frac{f''(z)}{z^{p-2}} - (p-1) \frac{f'(z)}{z^{p-1}} = \frac{p(\alpha-1)zw'(z)}{(1+w(z))^2}$$

or

$$(2.13) \quad \left| \frac{f''(z)}{z^{p-2}} - (p-1) \frac{f'(z)}{z^{p-1}} \right|^\gamma = \frac{p^\gamma |zw'(z)|^\gamma |1 - \alpha|^\gamma}{|1 + w(z)|^{2\gamma}}.$$

From (2.12) and (2.13), it follows that

$$\left| \frac{f'(z)}{pz^{p-1}} - 1 \right|^\beta \left| \frac{f''(z)}{z^{p-2}} - (p-1) \frac{f'(z)}{z^{p-1}} \right|^\gamma = \frac{p^\gamma |w(z)|^\beta (1 - \alpha)^{\beta+\gamma} |zw'(z)|^\gamma}{|1 + w(z)|^{\beta+2\gamma}}.$$

Suppose there exists a point $z_0 \in \mathbb{D}$ such that

$$|w(z_0)| = 1 \text{ and } |w(z)| < 1 \text{ when } |z| < |z_0|.$$

Then (2.4) and Lemma 2.1 yield

$$\begin{aligned} \left| \frac{f'(z_0)}{pz_0^{p-1}} - 1 \right|^\beta \left| \frac{f''(z_0)}{z_0^{p-2}} - (p-1) \frac{f'(z_0)}{z_0^{p-1}} \right|^\gamma &= \frac{p^\gamma (1 - \alpha)^{\beta+\gamma} |w(z_0)|^\beta |mw(z_0)|^\gamma}{|1 + e^{i\theta}|^{\beta+2\gamma}} \\ &= \frac{p^\gamma m^\gamma (1 - \alpha)^{\beta+\gamma}}{(2 + 2 \cos \theta)^{(\beta+2\gamma)/2}} \\ &\geq \frac{p^\gamma n^\gamma (1 - \alpha)^{\beta+\gamma}}{2^{\beta+2\gamma}}, \end{aligned}$$

which contradicts the hypothesis (2.9). Hence $|w(z)| < 1$, which implies

$$\left| \frac{1 - \frac{f'(z)}{pz^{p-1}}}{\frac{f'(z)}{pz^{p-1}} - \alpha} \right| < 1,$$

or equivalently

$$\operatorname{Re} \left(\frac{f'(z)}{pz^{p-1}} \right) > \frac{1 + \alpha}{2}.$$

For the second implication in the proof, consider the function w defined by

$$(2.14) \quad \frac{f'(z)}{pz^{p-1}} = 1 + (1 - \alpha)w(z).$$

Then

$$(2.15) \quad \left| \frac{f'(z)}{pz^{p-1}} - 1 \right|^\beta = |1 - \alpha|^\beta |w(z)|^\beta$$

and

$$(2.16) \quad \left| \frac{f''(z)}{z^{p-2}} - (p-1) \frac{f'(z)}{z^{p-1}} \right|^\gamma = p^\gamma |zw'(z)|^\gamma |1 - \alpha|^\gamma.$$

From (2.15) and (2.16), it is clear that

$$\left| \frac{f'(z)}{pz^{p-1}} - 1 \right|^\beta \left| \frac{f''(z)}{z^{p-2}} - (p-1) \frac{f'(z)}{z^{p-1}} \right|^\gamma = p^\gamma |w(z)|^\beta |1 - \alpha|^{\beta+\gamma} |zw'(z)|^\gamma.$$

Suppose there exists a point $z_0 \in \mathbb{D}$ such that

$$|w(z_0)| = 1 \text{ and } |w(z)| < 1 \text{ when } |z| < |z_0|.$$

Then by applying Lemma 2.1 and using (2.4), it follows that

$$\begin{aligned} \left| \frac{f'(z_0)}{pz_0^{p-1}} - 1 \right|^\beta \left| \frac{f''(z_0)}{z_0^{p-2}} - (p-1) \frac{f'(z_0)}{z_0^{p-1}} \right|^\gamma &= p^\gamma |w(z_0)|^\beta |1 - \alpha|^{\beta+\gamma} |z_0 w'(z_0)|^\gamma \\ &= p^\gamma m^\gamma |1 - \alpha|^{\beta+\gamma} \\ &\geq (pn)^\gamma |1 - \alpha|^{\beta+\gamma}, \end{aligned}$$

which contradicts the hypothesis (2.10). Hence $|w(z)| < 1$ and this implies

$$\left| \frac{f'(z)}{pz^{p-1}} - 1 \right| < 1 - \alpha$$

. Thus the proof is complete. \square

In next theorem, we need the concept of subordination. Let f and g be analytic functions defined on \mathbb{D} . Then f is *subordinate* to g , written $f \prec g$, provided there is an analytic function $w : \mathbb{D} \rightarrow \mathbb{D}$ with $w(0) = 0$ such that $f = g \circ w$.

THEOREM 2.4. *Let λ_1 and λ_2 be given by*

$$\begin{aligned} \lambda_1 &= \frac{2n+4(2p-1)}{4+n-2p+\sqrt{16n+n^2+32p-12np-28p^2}}, \\ \lambda_2 &= \frac{2n+4(2p-1)}{-n+2p+\sqrt{16-8n+n^2-48p+4np+36p^2}}, \end{aligned}$$

and $\lambda_1 < \lambda < \lambda_2$. If the function $f \in \mathcal{A}_{p,n}$ satisfies the inequality

$$(2.17) \quad \operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) < \begin{cases} \frac{2(1-p)\lambda^2+(4+n)\lambda+(2-2p-n)}{2(\lambda+1)}, & \lambda_1 < \lambda \leq \frac{p+n}{p}; \\ \frac{2(1-p)\lambda^2+n\lambda+(-2+2p+n)}{2(\lambda-1)}, & \frac{p+n}{p} < \lambda < \lambda_2, \end{cases}$$

then

$$(2.18) \quad \frac{1}{p} \frac{zf'(z)}{f(z)} \prec \frac{\lambda(1-z)}{\lambda-z}.$$

PROOF. Let us define w by

$$(2.19) \quad \frac{1}{p} \frac{zf'(z)}{f(z)} = \frac{\lambda(1-w(z))}{\lambda-w(z)}.$$

By doing the logarithmic differentiation on (2.19), we get

$$1 + \frac{zf''(z)}{f'(z)} = \frac{p\lambda(1-w(z))}{\lambda-z} - \frac{zw'(z)}{1-w(z)} + \frac{zw'(z)}{\lambda-w(z)}.$$

Assume that there exists a point $z_0 \in \mathbb{D}$ such that $|w(z_0)| = 1$ and $|w(z)| < 1$ when $|z| < |z_0|$. By applying Lemma 2.1 as in Theorem 2.1, it follows that

$$\begin{aligned} \operatorname{Re} \left(1 + \frac{z_0 f''(z_0)}{f'(z_0)} \right) &= \operatorname{Re} \left(\frac{p\lambda(1-e^{i\theta})}{\lambda-e^{i\theta}} \right) - \operatorname{Re} \left(\frac{me^{i\theta}}{1-e^{i\theta}} \right) + \operatorname{Re} \left(\frac{me^{i\theta}}{\lambda-e^{i\theta}} \right) \\ &= \frac{p\lambda(\lambda+1)(1-\cos\theta)}{\lambda^2+1-2\lambda\cos\theta} + \frac{m}{2} + \frac{m(\lambda\cos\theta-1)}{\lambda^2+1-2\lambda\cos\theta} \\ &= \frac{\lambda+1}{2}(2-p) + \frac{(\lambda^2-1)[(p+m)-p\lambda]}{2(\lambda^2+1-2\lambda\cos\theta)} \\ &\geq \frac{\lambda+1}{2}(2-p) + \frac{(\lambda^2-1)[(p+n)-p\lambda]}{2(\lambda^2+1-2\lambda\cos\theta)}, \end{aligned}$$

which yields the inequality

$$(2.20) \quad \operatorname{Re} \left(1 + \frac{z_0 f''(z_0)}{f'(z_0)} \right) \geq \begin{cases} \frac{2(1-p)\lambda^2+(4+n)\lambda+(2-2p-n)}{2(\lambda+1)}, & \lambda_1 < \lambda \leq \frac{p+n}{p}; \\ \frac{2(1-p)\lambda^2+n\lambda+(-2+2p+n)}{2(\lambda-1)}, & \frac{p+n}{p} < \lambda < \lambda_2. \end{cases}$$

Since (2.20) obviously contradicts hypothesis (2.17), it follows that $|w(z)| < 1$. This proves the subordination (2.18). \square

REMARK 2.1. *The subordination (2.18) can be written in equivalent form as*

$$\left| \frac{\lambda(zf'(z)/f(z)-1)}{zf'(z)/f(z)-\lambda} \right| < 1,$$

or by further computation, as

$$\left| \frac{1}{p} \frac{zf'(z)}{f(z)} - \frac{\lambda}{\lambda+1} \right| < \frac{\lambda}{\lambda+1}.$$

The last inequality shows that f is starlike in \mathbb{D} .

REMARK 2.2. *When $p = 1$ and $n = 1$, Theorems 2.1–2.4 reduce to Theorems 1.1 and 1.2.*

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